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GROUP CLASSIFICATION OF EQUATIONS OF THE FORM $y'' = f(x, y)$

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The problem of classification of ordinary differential equations of the form $y'' = f(x, y)$ by admissible local Lie groups of transformations is solved. "Standard" equations are listed on the basis of the equivalence concept. The classes of equations admitting a one-parameter group and obtained from the "standard" equations by invariant extension are described.

Key words: *equivalence, admissible operators, invariant extension.*

Introduction. The problem of group classification of differential equations was first posed by Norwegian mathematician Sophus Lie, the founder of the theory of continuous groups [1]. He also began to solve the problem of group classification of the second-order ordinary equation $y'' = f(x, y, y')$ and proved that this equation admits no more than an eight-parameter transformation group of the space $\mathbb{R}^2(x, y)$ and the maximum is reached if and only if this equation is equivalent to the linear equation $y'' = \varphi(x)y' + \psi(x)y + \omega(x)$. In the present paper, the problem of group classification of such equations is solved in the simpler case where the right side does not depend on the first derivative. This condition appears very stringent and leads to a relatively small list of the possible forms of the equations.

The solution of the group classification problem is related to the concept of equivalence of equations of this form with respect to transformations. We consider smooth, locally one-to-one maps (transformations) e: $(x, y, f) \to (x_1, y_1, f_1)$ of the space $\mathbb{R}^3(x, y, f)$ that act by the formulas

$$x_1 = F(x, y), \qquad y_1 = G(x, y), \qquad f_1 = H(x, y, f) \tag{1}$$

and satisfy the condition

$$\frac{\partial(x_1, y_1, f_1)}{\partial(x, y, f)} \equiv (F_x G_y - F_y G_x) H_f \neq 0. \tag{2}$$

Definition 1. A map e (1), (2) is called an *equivalence transformation* (ET) of the equality $y'' = f$ if it transforms the equation

$$y'' = f(x, y) \tag{3}$$

to an equation of the same form (here $y_1'' = d^2y_1/dx_1^2$):

$$y_1'' = f_1(x_1, y_1). \tag{4}$$

In this case, Eqs. (3) and (4) and the functions $f(x, y)$ and $f_1(x_1, y_1)$ are called *equivalent*.

The influence of the ET concept on group classification is determined by the fact that *equivalent equations admit similar groups* [2] and ET is a similarity transformation. That is, if (3) admits the group G, then (4) admits a group similar to it $G_1 = e(G)$. It is clear that the indicated correspondence is a set-theoretical criterion of equivalence, according to which the set of equations of the form (3) is split into classes of equivalent equations. Therefore, the group classification problem reduces to the following two problems: (a) to describe the classes of equivalent equations; (b) to find the admissible group for any (simplest) representative of each class.

Obviously, all possible ETs form the group $E = \{e\}$ of transformations of the space $\mathbb{R}^3(x, y, f)$, which is called *the equivalence group* of equations of the form (3). To solve problem (a), it is first necessary to describe this group.

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1. Equivalence Group. Calculation of the expression for the derivative $y_1'' = d^2y_1/dx_1^2$ in the variables (x, y) obtained by the substitution (1) gives the relation

$$(F_x + y'F_y)^3 y_1'' = (F_x + y'F_y)(G_{xx} + 2y'G_{xy} + y'^2 G_{yy} + y''G_y)$$

$$- (G_x + y'G_y)(F_{xx} + 2y'F_{xy} + y'^2 F_{yy} + y''F_y).$$

Transformation of Eq. (3) to Eq. (4) by the substitution (1) is possible if and only if this relation does not contain terms with the first derivative y'. Therefore, it splits into powers of y', leading to the equalities

$$F_x^3 f_1 = Jf + F_x G_{xx} - G_x F_{xx}; \tag{1.1}$$

$$3F_x^2 F_y f_1 = F_y G_{xx} - G_y F_{xx} + 2F_x G_{xy} - 2G_x F_{xy}; \tag{1.2}$$

$$3F_x F_y^2 f_1 = F_x G_{yy} - G_x F_{yy} + 2F_y G_{xy} - 2G_y F_{xy}; \tag{1.3}$$

$$F_y^3 f_1 = F_y G_{yy} - G_y F_{yy}, \tag{1.4}$$

where $J = F_x G_y - F_y G_x \neq 0$.

By virtue of (2), from (1.4) it follows that $F_y = 0$ and, hence, $F = \alpha(x)$ and equalities (1.2) and (1.3) are simplified:

$$G_y F_{xx} = 2F_x G_{xy}, \qquad G_{yy} = 0. \tag{1.5}$$

In this case, Eq. (1.1) becomes

$$F_x^3 f_1 = F_x G_y f + F_x G_{xx} - G_x F_{xx}. \tag{1.6}$$

The general solution of system (1.5) is $G = \beta(x)y + \gamma(x)$ with $\alpha''\beta = 2\alpha'\beta'$ and $\alpha'\beta \neq 0$, where the last inequality follows from the condition $J \neq 0$. Substitution of the expressions for F and G into (1.6) gives the relation

$$\alpha'^3 f_1 = \alpha'\beta f + (\alpha'\beta'' - \alpha''\beta')y + (\alpha'\gamma'' - \alpha''\gamma'),$$

which is simplified by the substitution $y = (y_1 - \gamma)/\beta$ to

$$\frac{\alpha'^2}{\beta} f_1 = f - \Big(\frac{1}{\beta}\Big)'' y_1 + \Big(\frac{\gamma}{\beta}\Big)''.$$

Thus, we obtain the general ET

$$\begin{gathered} x_1 = \alpha(x), \qquad y_1 = \beta(x)y + \gamma(x), \qquad \alpha''\beta = 2\alpha'\beta' \qquad (\alpha'\beta \neq 0), \\ (\alpha'^2/\beta) f_1 = f - (1/\beta)'' y_1 + (\gamma/\beta)'', \end{gathered} \tag{1.7}$$

which depends on two arbitrary functions, $\beta(x)$ and $\gamma(x)$, and two arbitrary constants which arise from the calculation of the function $\alpha(x)$.

For the further consideration, it is useful to note some particular forms of ETs.

Lemma 1. *Let $f_0(x, y)$ be a fixed function. Then,*

(i) *The function $f(x, y) = Af_0(Bx+C, My+N)$ with constants A, B, C, M, N is equivalent to the function $f_1(x_1, y_1) = (AM/B^2) f_0(x_1, y_1)$ for $B \neq 0$ or $f_1(y_1) = AMf_0(y_1)$ for $B = 0$ [$f_0 = f_0(y)$];*

(ii) *The function $f(x, y) = f_0(x, y) + p(x)y + q(x)$ is equivalent to the function $f_1(x_1, y_1) = A(x_1) f_0(x_1, y_1)$, where $A(x_1) = (\beta/\alpha'^2)(x_1)$; the functions α and β are obtained from the equations $\beta(1/\beta)'' = p(x)$ and $\beta\alpha'' = 2\beta'\alpha'$, and the dependence of x on x_1 is obtained by inversion of the function $x_1 = \alpha(x)$;*

(iii) *The function $f_0(x, y)$ is equivalent to the function $f_1(x_1, y_1) = x_1^{-3} f_0(1/x_1, y_1/x_1)$.*

Proof. All statements follow from the ET (1.7). In case (i) with $B \neq 0$ the ET $x_1 = Bx + C$, $y_1 = My + N$ is used, and for $B = 0$, the ET $x = x_1$, $y_1 = My + N$ is used. In case (ii), the right side of expression (1.7) for f_1 by virtue of the equality $y_1 = \beta y + \gamma$ is brought to the form

$$f - [(1/\beta)'' - p/\beta]y_1 + [(\gamma/\beta)'' - p\gamma/\beta + q]$$

and the choice of β and γ as solutions of the equations

$$(1/\beta)'' = p/\beta, \quad (\gamma/\beta)'' = p\gamma/\beta - q$$

leads to relation (1.7) in the required form $(\alpha'^2/\beta)f_1 = f_0$. Case (iii) is obtained from (1.7) with the functions $\alpha = 1/x$, $\beta = 1/x$, and $\gamma = 0$.

In particular, if one sets $f_0 \equiv 0$, then in case (ii) the function f is linear in y and is equivalent to the function $f_1 = 0$.

2. Admissible Operators. The operators of the one-parameter subgroups admitted by Eq. (3) are sought in the form

$$X = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y. \tag{2.1}$$

The standard algorithm for calculating such operators (see [2]) leads to the following determining equations (DEs):

$$\xi_{yy} = 0, \qquad \eta_{yy} = 2\xi_{xy}, \qquad 3f\xi_y = 2\eta_{xy} - \xi_{xx}; \tag{2.2}$$

$$\eta_{xx} + (\eta_y - 2\xi_x)f = \xi f_x + \eta f_y. \tag{2.3}$$

Here the function $f = f(x,y)$ is the right side of (3).

If the function f is linear in y, i.e., if Eq. (3) is a linear equation, this function is equivalent to the function $f \equiv 0$. In this case, system (2.2), (2.3) has the general solution

$$\xi = a(x)y + b(x), \qquad \eta = a'(x)y^2 + c(x)y + d(x),$$

where $a'' = c'' = d'' = 0$ and $b'' = 2c'$. This gives the well-known eight-dimensional admissible Lie algebra of operators.

Next, it is assumed that $f_{yy} \neq 0$. Then, the third of Eqs. (2.2) results in the equalities $\xi_y = 0$ and $2\eta_{xy} = \xi_{xx}$. In this case, the subsystem (2.2) is easily integrated and its general solution is $\xi = a(x)$, $\eta = b(x)y + c(x)$, where $2b' = a''$.

Thus, the admitted operators (2.1) have the form

$$X = a(x)\,\partial_x + [b(x)y + c(x)]\,\partial_y, \qquad a'' = 2b' \tag{2.4}$$

and the DE (2.3), namely,

$$b''y + c'' + (b - 2a')f = af_x + (by + c)f_y \tag{2.5}$$

remains, which is used to classify the nonlinear equations (3).

First, it should be noted that for the admissible operators, the inequality $a \neq 0$ should necessarily hold. Indeed, if $a = 0$, then $b = \text{const}$. Then, differentiation of the DE (2.5) with respect to y yields the relation $(by + c)f_{yy} = 0$, and for $f_{yy} \neq 0$, we have $b = c = 0$, i.e., the operator X is zeroth.

Next, we use the main property of ETs: if an ET transforms Eq. (3) to (4), the ET transforms an admissible group for (3) to an admissible group for (4).

Let (2.4) be a fixed admissible operator. Under the action of the ET (1.7), this operator is converted to an operator of the same form $[X_1 = a_1\,\partial_{x_1} + (b_1y_1 + c_1)\,\partial_{y_1}]$ with the coordinates

$$a_1 = a\alpha', \qquad b_1 = a\beta'/\beta + b, \qquad c_1 = a\gamma' - b_1\gamma + c\beta, \tag{2.6}$$

which should be considered as functions of x_1.

Lemma 2. *The following properties are valid:*

(i) *the constant* $a' - 2b = n$ *is an invariant of any ET*;

(ii) *if* $n = 0$, *the operator* (2.4) *is equivalent to the operator* $X_1 = \partial_x$;

(iii) *if* $n \neq 0$, *the operator* (2.4) *is equivalent to the operator* $X_1 = nx\partial_x$.

Proof. (i) Relations (2.6) imply the identity

$$\frac{da_1}{dx_1} - 2b_1 = \frac{da}{dx} - 2b + a\Big(\frac{\alpha''}{\alpha'} - 2\,\frac{\beta'}{\beta}\Big), \tag{2.7}$$

in which the expression in parentheses is equal to zero by virtue of (1.7);

(ii) By virtue of (2.7), it is possible to find an ET of the form (2.6) such that $a_1 = 1$ and $b_1 = c_1 = 0$;

(iii) Similarly, an ET of the form (2.6) exists such that $a_1 = nx_1$ and $b_1 = c_1 = 0$.

Corollary 1. If Eq. (3) is admitted by a certain operator of the form (2.4), it is equivalent to an equation of the same form that admits the operator $X_1 = \partial_x$ or the operator $X_1 = x\,\partial_x$.

3. Group Classification. The classification is performed by the following scheme.

Step s_1: the general form of the function $f(x, y)$ admitting the operators $X_1 = \partial_x$ or $X_1 = x\,\partial_x$ is established;

Step s_2: the expression obtained for f is substituted into the general DE (2.5) and the latter is used to find all functions f that provide the extension of the one-dimensional subalgebra with the operator X_1 to the complete admissible Lie algebra of operators.

First Possibility: $X_1 = \partial_x$. Step s_1 yields $f = f(y)$, and step s_2 yields DE

$$b''y + c'' + (b - 2a')f(y) = (by + c)f'(y). \tag{3.1}$$

Double differentiation with respect to y gives the equation

$$(b + 2a')f'' + (by + c)f''' = 0. \tag{3.2}$$

If $f''' = 0$, then f is equivalent (Lemma 1) to the function $f = y^2$. Substitution into (3.1) gives the DE

$$b''y + c'' + (b - 2a')y^2 = 2y(by + c),$$

which is split into powers of y taking into account relation (2.4) to give to the following expressions (a_1 and a_0 are constants):

$$a = a_0 + a_1 x, \qquad b = -2a_1, \qquad c = 0.$$

Hence, Eq. (3) with $f = y^2$ admits, in addition to $X_1 = \partial_x$, only one operator $X_2 = x\,\partial_x - 2y\,\partial_y$.

In the case $f''' \neq 0$, Eq. (3.2) implies the equality $(f''/f''')'' = 0$, whence it follows that

$$f'''/f'' = 1/(Ay + B) \tag{3.3}$$

with some constants A and B. Here two subcases are possible: $A \neq 0$ or $A = 0$ and $B \neq 0$.

If $A \neq 0$, then integration of (3.3) and the condition $f'' \neq 0$ give the functions f which are equivalent (Lemma 1) to one of the following functions:

$$f = Ny^k \quad (k \neq 0, 1, 2), \qquad f = N \ln y, \qquad f = Ny \ln y,$$

where $N = \text{const}$. Substitution into (3.1) shows that the last two functions do not give new operators. The DE (3.1) with the function $f = Ny^k$ becomes

$$b''y + c'' + (b - 2a')Ny^k = (by + c)kNy^{k-1}.$$

Splitting into powers of y gives the relations

$$2a' + (k - 1)b = 0, \qquad b'' = 0, \qquad c = 0.$$

The first of them, by virtue of (2.4), leads to the equality

$$(k + 3)b' = 0$$

and entails the alternative: $b' = 0$ or $k = -3$. If $b' = 0$, i.e., $b = 2b_1 = \text{const}$, we have $a' = -(k - 1)b_1$ and $a = a_0 - (k - 1)b_1 x$. The constant b_1 gives one additional operator $X_2 = (k - 1)x\,\partial_x - 2y\,\partial_y$, which was already obtained above in the case $k = 2$. If $k = -3$, the previous relations lead to the expressions $a = a_0 + 2b_0 x + b_1 x^2$ and $b = b_0 + b_1 x$ and the constants b_0 and b_1 give two additional operators

$$X_2 = 2x\,\partial_x + y\,\partial_y, \qquad X_3 = x^2\,\partial_x + xy\,\partial_y.$$

In the subcase $A = 0$, integration of Eq. (3.3) leads, with accuracy up to the ET, to the function $f = e^y$. With this function, Eq. (3.1) implies the equations $b = 0$, $2a' = -c$, and $c'' = 0$, which, together with (2.4), have the general solution

$$a = a_0 + a_1 x, \qquad b = 0, \qquad c = -2a_1.$$

The constant a_1 gives the additional admissible operator $X_2 = x\,\partial_x - 2\,\partial_y$.

Second Possibility: $X_1 = x\,\partial_x$. Step s_1 gives $f = x^{-2}g(y)$ with an arbitrary function $g(y)$, and at step s_2, we obtain the DE

$$x^3(b''y + c'') + [2(a - xa') + xb]g(y) = x(by + c)g'(y). \tag{3.4}$$

Equation (3.4) is analyzed similarly to (3.1). In this case, only one function, which is equivalent to $g(y) = y^{-1}$, is distinguished; with this function, one additional operator $X_2 = x^2\,\partial_x + xy\,\partial_y$ is admitted. However, the equation $y'' = x^{-2}y^{-1}$ is equivalent to the equation $y'' = y^{-1}$ by virtue of Lemma 1 [see (iii)].

TABLE 1

f	X_1	X_2	X_3
$f(y)*$	∂_x	0	0
e^y	∂_x	$x\,\partial_x - 2\,\partial_y$	0
$y^k,\ k \neq -3$	∂_x	$(k-1)x\,\partial_x - 2y\,\partial_y$	0
$\pm\, y^{-3}$	∂_x	$2x\,\partial_x + y\,\partial_y$	$x^2\,\partial_x + xy\,\partial_y$
$x^{-2}g(y)*$	$x\,\partial_x$	0	0

The results of the group classification of the nonlinear equations (3) are presented in Table 1, where the first column gives paired nonequivalent forms of the function f (the asterisk indicates that the function is arbitrary), and the basis operators of the admissible Lie algebra are listed in the next three columns.

4. Invariant Extensions. When using Table 1, one should take into account the following. Since for all operators from this table, the equalities $b'' = c'' = 0$ are valid, then the DE (2.5) for the admissible operators (2.4) can be written as

$$(b - 2a')f = X(f). \tag{4.1}$$

From this it follows that if Eq. (3) with the function $f_0(x,y)$ admits any operator X and $I = I(x,y)$ is any invariant of this operator, then Eq. (3) with the function $f = f_0 I$ also admits the same operator X. This follows from the fact that $X(I) = 0$, hence, $X(f_0 I) = X(f_0)I$, and from Eq (4.1).

This operation can be called an *invariant extension* of the sets of equations that admit even one of the operators listed in Table 1.

For example, the operator X_2 with the invariant $I_0 = x^2 y^{k-1}$ extends the equation $y'' = y^k$ $(k \neq -3)$ to the family of equations $y'' = y^k I_0^m = x^{2m} y^{k+m(k-1)}$, i.e., $y'' = x^p y^q$, which admit X_2, where

$$p = 2m, \qquad q = k(m+1) - m \tag{4.2}$$

can be any real numbers.

However, it should be taken into account that such extension can lead to nonequivalent (in the sense of the definition of ETs given in the introduction) equations. In particular, the equation $y'' = y^k$ is nonequivalent to the equation $y'' = x^p y^q$ with the exponents (4.2) for $m \neq 0$ because they admit nonsimilar groups.

Nevertheless, the invariant extension operation is useful in applications to particular problems because if Eq. (3) admits even one operator, it reduces to a first-order equation and a quadrature.

In conclusion, it is pertinent to note that the analysis performed in the present paper does not give a complete solution of the equivalence problem, which consists of establishing *the equivalence criterion for a priori given Eqs.* (3) and (4). Such criterion can be obtained only using the theory of differential invariants.

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